The Fundamental Group of a Co-H-Space is Free

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The goal of this note is to provide a proof of the following statement, which was originally set as an (admittedly difficult) exercise. We mainly follow Arkowitz's presentation [1].

Proposition 0.1 *Let (X, c) be a co-H-space. Assume that X is well-pointed. Then $\pi_1 X$ is a free group.*

The proof follows from a pair of more general lemmas. The idea is that the functor π_1 converts the topological comultiplication c on X into an algebraic comultiplication on $\pi_1 X$. To make sense of this we need the key assumption of well-pointedness, which allows us to apply the Seifert-van Kampen Theorem to the pushout defining the wedge.

Lemma 0.2 *The inclusions induce an isomorphism $\pi_1 X * \pi_1 X \xrightarrow{\cong} \pi_1(X \vee X)$.* $\blacksquare$

The form of the isomorphism allows us to conclude that the diagram

$$\begin{array}{ccc} \pi_1 X & \xrightarrow{c_*} & \pi_1 X * \pi_1 X \\ & \searrow^{D} & \downarrow \\ & & \pi_1 X \times \pi_1 X \end{array} \tag{0.1}$$

commutes, where D is the diagonal homomorphism of $\pi_1 X$. The point is that the induced homomorphism c_* is a coproduct on $\pi_1 X$ in the category of groups. As it turns out, the only groups which admit such operations are the free groups [3].

Having seen how the functor π_1 converts the topological problem into an algebraic one, let us now turn to address the algebraic side of the story. Let G, H be groups and $\varphi, \phi : G \to H$ a pair of homomorphisms. The equaliser of φ, ϕ is the subgroup

$$Eq(\varphi, \phi) = \{g \in G \mid \varphi(g) = \phi(g)\} \leq G. \tag{0.2}$$

We also need to introduce notation for the free product $G * G$. The free product is the coproduct in the category of groups, so there are a pair of homomorphisms

$$G \xrightarrow{\iota_1} G * G \xleftarrow{\iota_2} G \tag{0.3}$$

and homomorphisms $G * G \to H$ are determined completely by their restrictions along ι_1, ι_2. If $g \in G$ we write $g' = \iota_1(g)$ and $g'' = \iota_2(g)$ for its images in $G * G$. Then any element $\xi \in G * G$ can be written as a word of the form $\prod_{i=1}^{n} g_i' h_i''$, where $g_i, h_i \in G$ and $n \geq 0$ is a finite integer which we'll call the **length** of ξ.

Lemma 0.3 *Let G be a group and denote by $q_1, q_2 : G * G \to G$ the homomorphism pinching to each of the respective factors. Then the equaliser of the pair*

$$G * G \underset{q_2}{\overset{q_1}{\rightrightarrows}} G \tag{0.4}$$

is free with basis $\Xi = \{g' \cdot g'' \mid g \in G\}$.

Proof Clearly the subgroup $\langle \Xi \rangle$ generated by Ξ is contained inside $Eq(q_1, q_2)$. We show by induction on word length that $\langle \Xi \rangle$ is equal to $Eq(q_1, q_2)$. Thus let $\xi = \prod_{i=1}^{n} g_i' h_i'' \in Eq(q_1, q_2)$. Clearly if $n = 1$, then $\xi \in \langle \Xi \rangle$. So assume that any word of length $< n$ that belongs to $Eq(q_1, q_2)$ lies in $\langle \Xi \rangle$ and let ξ be as above. Consider the element

$$\rho = g_1''^{-1} g_1'^{\,-1}(\xi) h_n''^{-1} h_n'^{\,-1} = (g_1''^{-1} h_1'') g_2' \dots g_{n-1}'(h_{n-1}'' h_n''^{-1}). \tag{0.5}$$

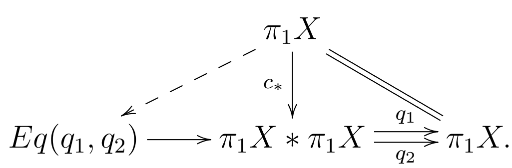
Clearly ρ^{-1} has length $< n$, so $\rho \in \langle \Xi \rangle$. But clearly this implies that ξ is expressible in terms of the elements in Ξ. Appealing to the inductive hypothesis we complete the proof ∎

We'll also need one technical lemma about free groups.

Lemma 0.4 *A subgroup of a free group is free.* ∎

Proof See Robinson [4] §6.1 for the standard proof. A topological proof can be found in Hatcher [2] §1.A pg. 94. ∎

Proof of 0.1 Consider the diagram



(0.6)

Following the discussion around (0.1) we have that $q_1 c_* = q_2 c_* = id_{\pi_1 X}$, so the induced homomorphism c_* is injective and factors through the equaliser $Eq(q_1, q_2)$. According to Lemma 0.3 $Eq(q_1, q_2)$ is free, and since $\pi_1 X$ embeds into it, it is also free. ∎

References

[1] M. Arkowitz, *Introduction to Homotopy Theory*, Springer, (2011).

[2] A. Hatcher, *Algebraic Topology*, Cambridge University Press, (2002). Available at http://pi.math.cornell.edu/ hatcher/AT/ATpage.html.

[3] B. Eckmann, P. Hilton, *Structure Maps in Group Theory*, Fund. Math. **50** (1961), 207-221.

[4] D. Robinson, *A Course in the Theory of Groups*, Springer, (1996).